Section 1.4

Rates of Change

Reminder: The Average Rate of Change formula between $x=a$ and $x=b$, for a function $f(x)$ is:

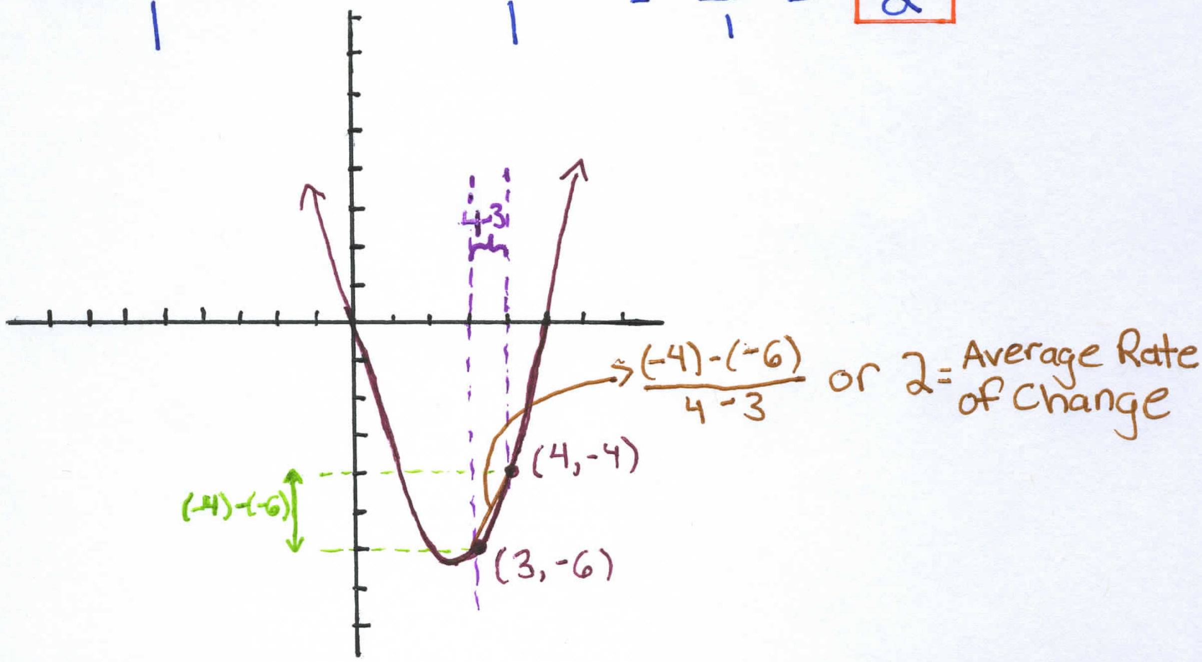
$$\frac{f(b)-f(a)}{b-a}$$

1. $f(x)=x^2-5x$ between $x=3$ and $x=4$.

$f(4)=(4)^2-5(4)=16-20=-4$ ...... so $f(4)=-4$

$f(3)=(3)^2-5(3)=9-15=-6$ ...... so $f(3)=-6$

$$\frac{f(4)-f(3)}{4-3}=\frac{(-4)-(-6)}{1}=\frac{-4+6}{1}=\frac{2}{1}=\boxed{2}$$

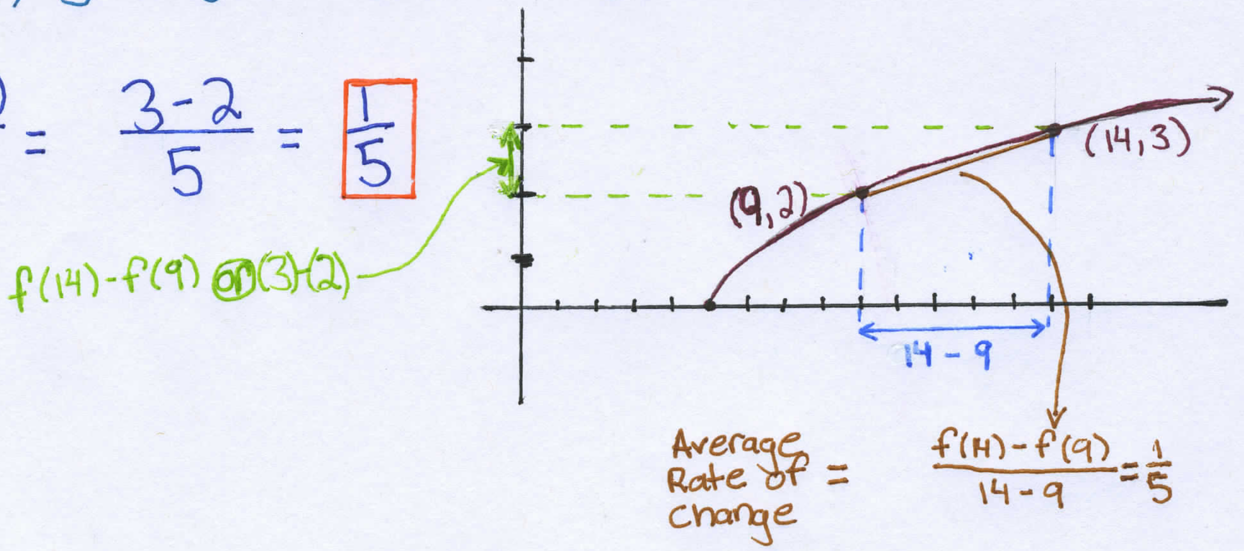


3. $f(x)=\sqrt{x-5}$ between $x=9$ and $x=14$.

$f(14)=\sqrt{(14)-5}=\sqrt{9}=3$ ...... so $f(14)=3$

$f(9)=\sqrt{(9)-5}=\sqrt{4}=2$ .... so $f(9)=2$

$$\frac{f(14)-f(9)}{14-9}=\frac{3-2}{5}=\boxed{\frac{1}{5}}$$



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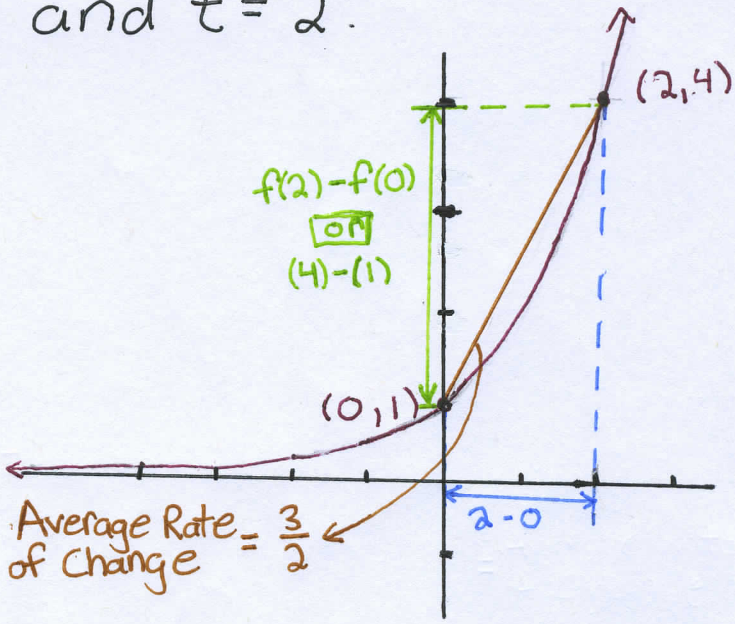
5. $s(t)=2^t$ between $t=0$ and $t=2$.

$s(2)=2^2=4$ ..... so $s(2)=4$

$s(0)=2^0=1$ ...... so $s(0)=1$

$$\frac{s(2)-s(0)}{2-0}=\frac{4-1}{2}=\boxed{\frac{3}{2}}$$



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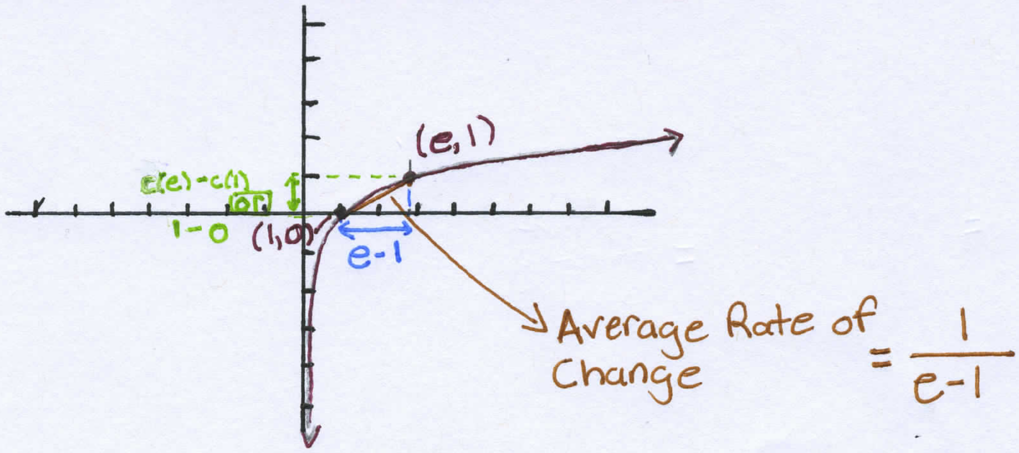
7. $c(t) = \ln(t)$ between $t=1$ and $t=e$.

$c(e) = \ln(e) = 1$ ...... so $c(e) = 1$

$c(1) = \ln(1) = 0$ ..... so $c(1) = 0$

$$\frac{c(e) - c(1)}{e-1} = \frac{1-0}{e-1} = \boxed{\frac{1}{e-1} \cong .5819767}$$



9. A climber is on a hike. After 2 hours he is at an altitude of 400 feet. After 6 hours, he is at an altitude of 700 feet. What is the average rate of change?

Note: Time is almost always the independent variable (the x in an f(x) function).

2 hr. → 400 ft. .... so $f(2)=400$

6 hr. → 700 ft. .... so $f(6)=700$

$$\frac{f(6)-f(2)}{6-2} = \frac{700-400}{4} = \frac{300}{4} = \boxed{75\ \text{ft}/\text{hr}}$$

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11. A rocket is 1 mile above the earth in 30 seconds and 5 miles above the earth in 2.5 minutes. What is the rockets average rate of change in miles per second?

Note: The question asks for an answer in seconds and one of the units is in minutes so we will have to convert it into seconds.

Ratio of Minutes and Seconds:	1 min : 60 sec OR $\frac{1\text{ min}}{60\text{ sec}}$ OR $\frac{60\text{ sec}}{1\text{ min}}$

$$2.5\cancel{\text{min}}\left(\frac{60\text{ sec}}{1\ \cancel{\text{min}}}\right) = 150\text{ sec}$$

$$\frac{f(150)-f(30)}{150-30} = \frac{5-1}{120} = \frac{4}{120}$$

30 sec. → 1 mile ...... So $f(30)=1$

150 sec. → 5 mile ...... So $f(150)=5$

$$\frac{4}{120} = \boxed{\frac{1}{30}\text{ miles per second}}$$

13.

Students may skip problem 13 due to phrasing issues.

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15. A plane left Chicago at 8:00 A.M., at 1:00 P.M., the plane landed in Los Angeles, which is 1500 miles away. What was the average speed of the plane for the trip?

Note: We are only given one distance, which is the total distance between Chicago and Los Angeles. Thinking in terms of the Average Rate of Change Formula this is the value we would need in the numerator, so $f(1\text{PM}) - f(8\text{AM}) = 1500$.

There is 5 hours between 8:00 A.M. and 1:00 P.M..

$$\frac{f(1\text{PM}) - f(8\text{AM})}{1\text{PM} - 8\text{AM}} = \frac{1500 \text{ miles}}{5 \text{ hours}} = \boxed{300 \text{ miles per hour}}$$

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17. $f(x) = x^2 - 3$ at $x = 2$

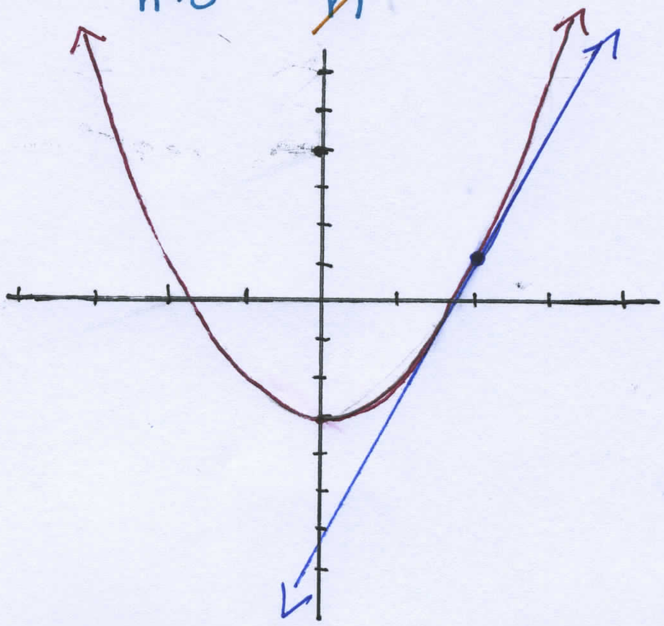
$f(2) = (2)^2 - 3 = 4 - 3 = \underline{1}$

$f(2+h) = (2+h)^2 - 3 = (2+h)(2+h) - 3 = \boxed{4} + 4h + h^2 \boxed{-3}$

$\underline{f(2+h) = h^2 + 4h + 1}$

$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{h^2 + 4h \cancel{+1} \cancel{-1}}{h} = \lim_{h \to 0} \frac{\cancel{h}(h+4)}{\cancel{h}}$

$\lim_{h \to 0} h + 4$ Thus: $(0) + 4 = 4$



Instantaneous Rate of change at $x = 2$ is $\underline{4}$.

19. $f(x) = 3 - x^2$ at $x = 1$

$f(1) = 3 - (1)^2 = 3 - 1 = \underline{2}$

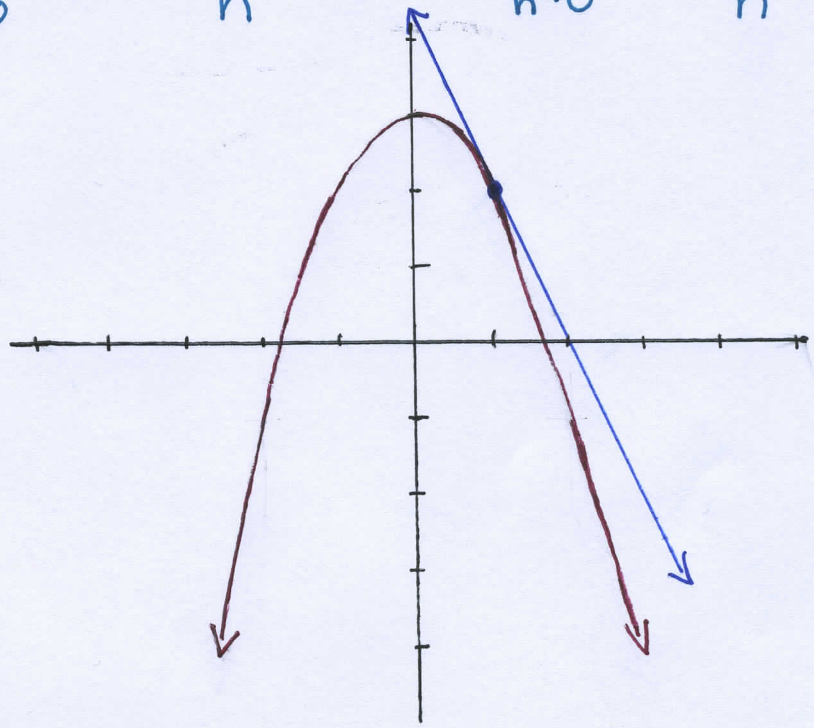
$f(1+h) = 3 - (1+h)^2 = 3 - (1+h)(1+h) = 3 - (1 + 2h + h^2)$

$= 3 - 1 - 2h - h^2 = \underline{-h^2 - 2h + 2}$

$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{-h^2 - 2h + 2 - 2}{h} = \lim_{h \to 0} \frac{h(-h-2)}{h}$

$= \lim_{h \to 0} -h - 2$

$-(0) - 2 = -2$



$\boxed{\text{Instantaneous Rate of Change when } x = 1 \text{ is } \underline{-2}.}$

21. $g(t) = t^2 + 2t - 3$ at $t = -2$

$$g(-2) = (-2)^2 + 2(-2) - 3 = \cancel{4} - \cancel{4} - 3 = \underline{-3}$$

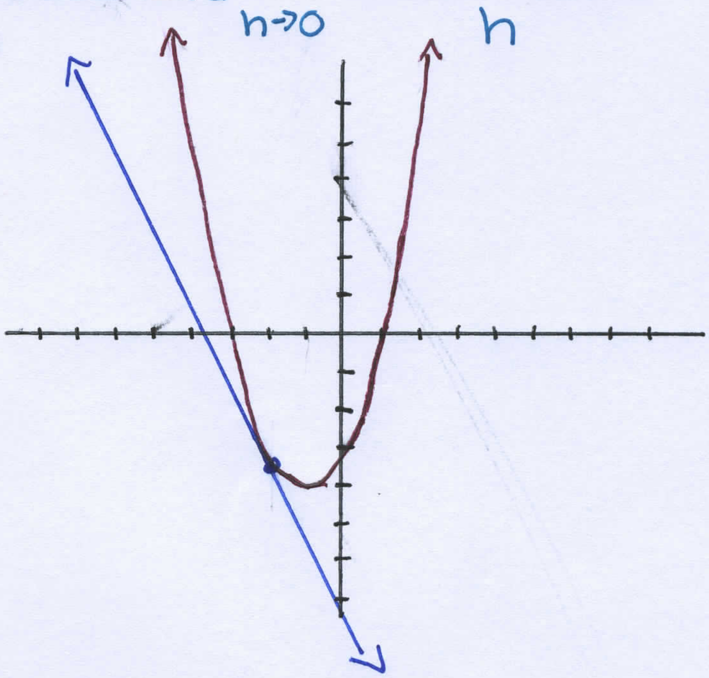
$$g(-2+h) = (-2+h)^2 + 2(-2+h) - 3 = \cancel{4} \boxed{-4h} + h^2 \cancel{-4} + \boxed{2h} - 3$$

$$\underline{g(-2+h) = h^2 - 2h - 3}$$

$$\lim_{h \to 0} \frac{g(-2+h) - g(-2)}{h} = \lim_{h \to 0} \frac{(h^2 - 2h - 3) - (-3)}{h} = \lim_{h \to 0} \frac{h^2 - 2h \cancel{-3} \cancel{+3}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(h-2)}{\cancel{h}} = \lim_{h \to 0} h - 2$$

$$(0) - 2 = -2$$



Instantaneous Rate of Change at $t = -2$ is $\underline{\mathbf{-2}}$.

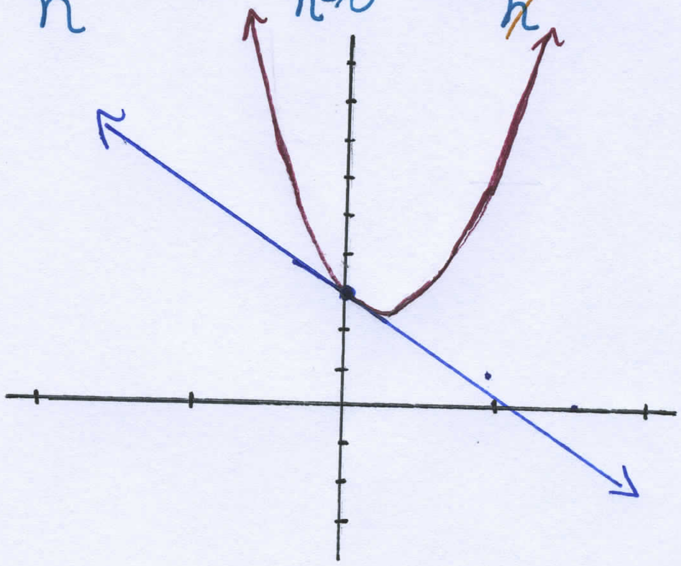
23. $h(t) = 5t^2 - 2t + 3$ at $t = 0$

$h(0) = 5(0)^2 - 2(0) + 3 = 0 - 0 + 3 = \underline{3}$

$h(0+h) = 5(0+h)^2 - 2(0+h) + 3 = 5(h)^2 - 2(h) + 3$

$h(0+h) = \underline{5h^2 - 2h + 3}$

$$\lim_{h \to 0} \frac{h(0+h) - h(0)}{h} = \lim_{h \to 0} \frac{5h^2 - 2h + \cancel{3} - \cancel{3}}{h} = \lim_{h \to 0} \frac{\cancel{h}(5h-2)}{\cancel{h}}$$



$$\lim_{h \to 0} 5h - 2$$

$5(0) - 2 = 0 - 2 = -2$

Instantaneous Rate of Change at $t=0$ is $\underline{-2}$.

25. ... $s(t) = -2t^2 + 30t + 5$ ... at $t = 3$.

$$s(3) = -2(3)^2 + 30(3) + 5 = -2(9) + 90 + 5 = -18 + 95 = \underline{77}$$

$$s(3+h) = -2(3+h)^2 + 30(3+h) + 5 = -2(3+h)(3+h) + 90 + 30h + 5$$

$$s(3+h) = -2(9 + 6h + h^2) + 30h + 95 = -18 - 12h - 2h^2 + 30h + 95$$

$$s(3+h) = \underline{-2h^2 + 18h + 77}$$

$$\lim_{h \to 0} \frac{s(3+h) - s(3)}{h} = \lim_{h \to 0} \frac{-2h^2 + 18h + \cancel{77} - \cancel{77}}{h} = \lim_{h \to 0} \frac{\cancel{h}(-2h + 18)}{\cancel{h}}$$

$$\lim_{h \to 0} -2h + 18$$

$$-2(0) + 18 = \underline{18}$$

<u>Answer</u>: The velocity, or instantaneous rate of change, of the rocket at $t = 3$ is <u>18</u>.

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27. A pebble is dropped from a cliff, 50 m high. After t sec., the pebble is s meters above the ground, where $s(t) = 50 - 2t^2$. Calculate the instantaneous rate of change (velocity) of the ball at $t = 2$ seconds.

Note: "50 m high" was not underlined as an important point since its significance is in creating the s(t) function already given. (The function is of the height of the pebble; starting at 50 and decreasing each second.)

$$s(2) = 50 - 2(2)^2 = 50 - 2(4) = 50 - 8 = \underline{42}$$

$$s(2+h) = 50 - 2(2+h)^2 = 50 - 2(4 + 4h + h^2) = 50 - 8 - 8h - 2h^2$$

$$s(2+h) = \underline{-2h^2 - 8h + 42}$$

$$\lim_{h \to 0} \frac{s(2+h) - s(2)}{h} = \lim_{h \to 0} \frac{-2h^2 - 8h + \cancel{42} - \cancel{42}}{h} = \lim_{h \to 0} \frac{\cancel{h}(-2h - 8)}{\cancel{h}}$$

$$= \lim_{h \to 0} -2h - 8$$

$$-2(0) - 8 = \underline{-8}$$

$$\boxed{\text{Answer: The velocity, or instantaneous rate of change, of the pebble at } t = 2 \text{ seconds } \underline{-8 \text{ m/sec}}.}$$

29. The profit from sale of x car seats is given by the formula: $P(x) = 45x - 0.0025x^2 - 5000$

a) Find the profit from selling 800 car seats.

$x = 800$ car seats

$P(800) = 45(800) - 0.0025(800)^2 - 5000$

$= 36000 - 0.0025(640000) - 5000$

$= 36000 - 1600 - 5000 = \boxed{\$29400}$

29. b) $P(x) = 45x - 0.0025x^2 - 5000$

$P(800) = 45(800) - 0.0025(800)^2 - 5000$

$P(800) = \underline{36{,}000} - 0.0025(640{,}000) \underline{-5000} = 31{,}000 - 1{,}600$

$\underline{P(800) = 29{,}400}$

$P(800+h) = 45(800+h) - 0.0025(800+h)^2 - 5{,}000$

$= \underline{36{,}000} + 45h - 0.0025(800+h)(800+h) \underline{-5{,}000}$

$= 31{,}000 + 45h - 0.0025(640{,}000 + 1600h + h^2)$

$= \underline{31{,}000} + \underline{45h} - \underline{1{,}600} - \underline{4h} - 0.0025h^2 = \underline{-0.0025h^2 + 41h + 29400}$

$(P(800+h))$

$$\lim_{h\to 0} \frac{P(800+h) - P(800)}{h} = \lim_{h\to 0} \frac{(-0.0025h^2 + 41h + 29{,}400) - (29{,}400)}{h}$$

$$= \lim_{h\to 0} \frac{-0.0025h^2 + 41h + \cancel{29{,}400} - \cancel{29{,}400}}{h} = \lim_{h\to 0} \frac{\cancel{h}(-0.0025h + 41)}{\cancel{h}}$$

$$= \lim_{h\to 0} -0.0025h + 41 \rightarrow -0.0025(0) + 41 = \$41$$

Additional profit from selling the next (or 801st) car seat.

Possible way to check with calculator:

- Enter "45x - 0.0025x² - 5000" in "Y=" section
- Adjust "WINDOW"
  - My settings were: Xmin=0 ; Xmax=1000 ; Ymin=0 ; Ymax=1000
- Find "dy/dx" (usually in "CALC") and enter "800" for "X=".

31. a) $x = 30$ chairs

$$C(30) = (30)^2 + 40(30) + 800 = 900 + 1{,}200 + 800$$

$$C(30) = \boxed{\$2{,}900}$$

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31. b) $C(30) = 2{,}900$

$$C(30+h) = (30+h)^2 + 40(30+h) + 800$$

$$= (30+h)(30+h) + 1{,}200 + 40h + 800 = 900 + 60h + h^2 + 2{,}000 + 40h$$

$$= h^2 + 100h + 2{,}900 = C(30+h)$$

$$\lim_{h\to 0} \frac{C(30+h) - C(30)}{h} = \lim_{h\to 0} \frac{h^2 + 100h \cancel{+2{,}900} \cancel{-2{,}900}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(h+100)}{\cancel{h}} = \lim_{h\to 0} h + 100 \quad \boxed{\begin{array}{l}\text{when } h=0 \\ \text{for } h+100\end{array}} : (0) + 100 = 100$$

Possible way to find the answer with a calculator:

- Enter "$x^2 + 40x + 800$" in "Y="
- Adjust "WINDOW"
  - My settings were: Xmin=0; Xmax=100; Ymin=0; Ymax=10000
- Find the "dy/dx" for x="30".

The additional cost to produce the 31st chair is \$100. (The marginal cost for the next chair is \$100.)